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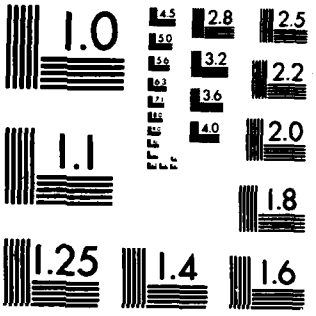

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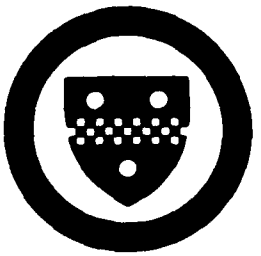

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ERROR ESTIMATES FOR NON-LINEAR FINITE ELEMENT COMPUTATIONS[1)]

by

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1. INTRODUCTION

In recent years, increasing interest has centered on the development of reliable and computationally inexpensive a posteriori error estimates for finite element computations, (see for example [1], [2], [3], [4], [5], [6], [7]). Such estimates can provide some, often critically important, information about the accuracy and reliability of the computed solution as a model of the behavior of the physical phenomena under study. At the same time it has become widely accepted that these estimates also constitute a basic tool in the construction of efficient adaptive finite element processes which are designed to achieve a desired error tolerance at minimal cost or a best possible solution within an allowable cost range, (see e.g. [8], [9], [10], [11], [12]).

Up to now most of this work concerned linear problems. Not unexpectedly, for non-linear problems the situation is much more difficult and the theory is by far not as well developed. In part this is due to the many special features of non-linear problems not present in the linear case. In particular, such problems usually involve a number of intrinsic parameters and -- because

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of the non-linear nature -- interest centers rarely on the determination of a few specific solutions for fixed parameter values but instead on a more general study of these solutions under various changes of the parameters. In structural mechanics the parameters may characterize, for instance, load points and load directions, material properties, geometrical data, etc., and the set of all solutions depending on these parameters has been called the equilibrium surface of the structure, [13]. This equilibrium surface provides considerable insight into the behavior of the structure and its stability properties, and the computational task is to analyze its shape and characteristic features. For this the principal tools are the continuation methods, or incremental methods as they are often called in the engineering literature. Generally, these processes allow for the trace of any path on the surface defined by a parameter combination with one degree of freedom.

In the analysis of a structural problem by the finite element method we are able to compute only approximate points along a path on the solution surface of some discretized form of the original problem. Then we are faced with the need for assessing and controlling the errors along an entire segment of such a path. In [14], and [15] it has been shown that effective, a posteriori error estimates and adaptive procedures can be constructed which meet these aims and which can be successfully incorporated into a general continuation process for tracing paths on the equilibrium surface. But these results were essentially restricted to problems in one space dimension, primarily because the estimates used there were computationally relatively expensive.

In this paper, we presents a new approach to the construction of

a posteriori error estimates for non-linear problems which is highly effective and at the same time computationally rather inexpensive. In fact, these estimates use a linearized form of the problem and hence their computation can be accomplished about as rapidly as in the case of linear problems. Moreover, the approach allows us to bring to bear most of the earlier cited results about linear a posteriori error estimates and hence it applies also to problems in more than one space dimension. Based on these estimates, a prototype software system for the adaptive finite element solution of a class of two-dimensional, parametrized non-linear problems is now under construction at the University of Pittsburgh. It was dubbed NFEARS, short for Nonlinear Finite Element Adaptive Research Solver, in analogy to an earlier developed system of the same type for linear problems called FEARS, (see [8], [9], [11], [16], [17]).

In Section 2 below we present the general principles behind the new error estimates. Then Section 3 outlines some of the features of the design of NFEARS. Finally in Section 4 we give some numerical results which show the effectiveness of the error estimates and of the adaptive approach.

2. A POSTERIORI ERROR ESTIMATES

As mentioned in the Introduction, the new error estimates for non-linear problems utilize the earlier developed estimation theory for linear problems. It would be impossible to present here an account of that theory; but it may be useful to illustrate the ideas on a simple example.

Consider the two-point boundary value problem

$$L[u] \equiv -\frac{\partial}{\partial s} a(s) \frac{\partial u}{\partial s} + b(s)u = c(s), \quad 0 < s < 1, \tag{2.1}$$

$$u(0) = u(1) = 0 \tag{2.2}$$

where the coefficient functions a, da/ds, b, and c are assumed to be continuous on $I = [0,1]$ and such that

$$0 < \alpha_0 \leq a(s) \leq \alpha_1 < \infty, \quad 0 \leq b(s) \leq \alpha_2 < \infty, \quad \forall\, s \in I. \tag{2.3}$$

It is well known that the bilinear form

$$B(u,v) = \int_I [a(s)u'v' + b(s)u,v]ds \tag{2.4}$$

is defined and continuous on $H_0^1(I) \times H_0^1(I)$, where in (2.4) primes denote derivatives with respect to s. Then the energy norm is given by

$$||v||_E = B(v,v)^{1/2}. \tag{2.5}$$

The weak solution of our problem is now the unique function $u_0 \in H_0^1(I)$ for which

$$B(u_0,v) = F(v) \equiv \int_I fvds, \quad \forall\, v \in H_0^1(I). \tag{2.6}$$

Suppose that we use piecewise linear elements on some mesh

$$\Delta: 0=s_0 < s_1 < s_2 < \ldots s_{n+1} = 1, \quad n = n(\Delta), \tag{2.7}$$

with not necessarily uniform steps $h_k = s_k - s_{k-1}$, $k = 1,\ldots,n+1$. In other words we restrict consideration to the finite dimensional subspace

$$S(\Delta) = \{u \in H_0^1(I), \quad u(s) = \sum_{i=1}^{n} x_i\, \phi_i(s); \quad s \in I\} \tag{2.8}$$

where for each $i = 1,\ldots,n$, ϕ_i denotes the continuous piecewise linear "hat function" which is 1 for $s = s_i$ and zero at all other nodes of Δ. Then the finite element solution $\bar{u}_0 \in S(\Delta)$ is uniquely defined by the condition

$$B(\bar{u}_0,v) = F(v), \quad \forall\ v \in S(\Delta).$$

In order to estimate the norm of the finite element error $e = u_0 - \bar{u}_0$ let P_k denote the orthogonal projection of $H_0^1(I)$ onto the subspace

$$\{v \in H_0^1(I);\ \ v(s) = 0, \ \forall\ s \notin (s_{k-1},s_k)\}$$

with respect to the scalar product defined by $B(u,v)$. Then with the error indicators

$$\eta_k = ||P_k e||_E, \quad k = 1,\ldots,n+1 \tag{2.9}$$

we can define the error estimator

$$\varepsilon = \Big(\sum_{k=1}^{n+1} \eta_k^2\Big)^{1/2}$$

for which it can be shown that $\varepsilon \le ||e||_E \le C\varepsilon$ with a constant $C < \infty$ that depends only on the bounds α_0, α_1, α_2 in (2.3) but not on a,b,c, and Δ. The restriction of the function $P_k e$ to $I_k = [s_{k-1},s_k]$ is the unique solution w_k of the original problem on I_k for which $w_k(s_{k-1}) = w_k(s_k) = 0$.

Obviously, we are interested in simple approximations of the w_k.

For instance, we may use the quadratic finite element approximation

$$\bar{w}_k(s) = \rho_k z_k(s), \quad z_k(s) = \frac{4}{h_k^2}(s-s_{k-1})(s_k-s), \quad s \in I_k.$$

Then we obtain the approximate error indicator

$$\bar{\eta}_k = ||\bar{w}_k|| = \frac{\{\int_{I_k} [a(s)\bar{u}_o' z_k' + b(s)\bar{u}_o z_k - c(s)z_k]ds\}^2}{\int_{I_k} [a(s)z_k' z_k' + b(s)z_k z_k]ds} \tag{2.10}$$

$$= \frac{\{\int_{I_k} r_k z_k \, ds\}^2}{\int_{I_k} [a(s)z_k' z_k' + b(s)z_k z_k]ds}$$

where $r_k(s) = L[\bar{u}_o](s) - c(s)$, $s \in I_k$, is the residual of $\bar{u}_o$ on I_k.

Evidently, there are various other forms of the error indicators and hence of the a posteriori error estimate. For more details and proofs we refer to [1], [3], [4].

In order to summarize the situation in the two-dimensional case, consider the problem defined by

$$\int_\Omega [(\nabla v)^T M(s,t)\nabla u - c(s,t)uv]ds\, dt = 0, \quad \forall\, v \in H_o^1(\Omega), \tag{2.11}$$

where $M(s,t)$ is a symmetric, positive definitive, 2×2-matrix for all $(s,t) \in \bar{\Omega} = [0,1] \times [0,1]$ and $\nabla u = (u_s, u_t)^T$ is the gradient.

Suppose that we use bilinear elements on a uniform mesh on $\bar{\Omega}$ with step $h > 0$. Then the evaluation of the residuals of the finite element

solution $\bar{u}_0$ results in a Dirac function with linear distribution on the sides of each element. More specifically, let P be any interior node of the mesh and denote its four horizontal and vertical neighbors by W,N,E,S in the sense of the windrose. If the corresponding function values are $\bar{u}_0(P)$, $\bar{u}_0(W)$, $\bar{u}_0(N)$, $\bar{u}_0(E)$, $\bar{u}_0(S)$, respectively, then the following jump values may be computed

$$JH(P) = \frac{1}{h}(\bar{u}_0(E)-\bar{u}_0(P)) - \frac{1}{h}(\bar{u}_0(P)-\bar{u}_0(W)),$$

$$JV(P) = \frac{1}{h}(\bar{u}_0(N)-\bar{u}_0(P)) - \frac{1}{h}(\bar{u}_0(P)-\bar{u}_0(S)).$$

At any point P on one of the vertical boundaries but not at a corner point of the domain, let P' be the immediate horizontal neighbor of P. Then we set $JH(P) = JH(P')$ and $JV(P) = 0$. Similarly, for any point P on a horizontal boundary but not at a corner of the domain we set $JV(P) = JV(P')$ and $JH(P) = 0$ where now P' is the immediate vertical neighbor of P. With these quantities the first part of the error indicator of any element τ of our mesh is given by

$$\eta_1(\tau) = \frac{h^2}{48}\alpha \sum_{P} [M_{11}(s,t)JH(P)^2 + M_{22}(s,t)JV(P)^2]. \tag{2.12}$$

Here M_{ij} denote the elements of the matrix M, the sum extends over all nodes P of τ which are not corners of $\bar{\Omega}$, and the factor α equals 4/3 if τ has a node at a corner of $\bar{\Omega}$ and 1, otherwise.

For the computation of the residual contribution to the error indicator let R be given by

$$R = [\frac{\partial}{\partial s} M_{12} + \frac{\partial}{\partial t} M_{22}]\frac{\partial}{\partial t} \bar{u}_o + [\frac{\partial}{\partial s} M_{11} + \frac{\partial}{\partial t} M_{21}]\frac{\partial}{\partial s} \bar{u}_o$$

$$+ [M_{12} + M_{21}]\frac{\partial^2}{\partial s \partial t} \bar{u}_o - c\, \bar{u}_o$$

where the M_{ij} are again the elements of the matrix H. Moreover, let R_j denote the values of R at the four Legendre-Gauss points of the element τ. Then the residual contribution to the error indicator of τ is

$$\eta_2(\tau) = \frac{h^4}{4\pi^2} \sum_{j=1}^{4} [(R_j - \frac{1}{4} (\sum_{j=1}^{4} R_j)]^2. \tag{2.13}$$

The total error indicator for the element τ is now

$$\eta(\tau) = \eta_1(\tau) + \eta_2(\tau)$$

and the sum of all the error indicators over the elements of the mesh is the square of the desired error estimate ε. For proofs we refer to [6].

In [14], and [15] the above indicated approach was applied directly to nonlinear problems. In that case, the auxiliary problems for the w_k become nonlinear and this is the source for the computational expense mentioned in the Introduction.

In order to reduce this expense we utilize here a basic property of the derivative of our nonlinear operators. In general, the parameterized equations under study have the form

$$F(y,\lambda) = 0$$

where y represents a state variable varying in an appropriate normed space, and λ is a finite dimensional parameter variable. For simplicity

we consider here only the case of a non-singular point $(\bar{u}_o,\lambda_o)$ where the derivative $D_uF(\bar{u}_o,\lambda_o)$ of F with respect to the state variable has a continuous inverse. At such a point the linearized operator has the form

$$L(\bar{u}_o,\lambda_o)[w] = F(\bar{u}_o,\lambda_o) + D_uF(\bar{u}_o,\lambda_o)(w-\bar{u}_o).$$

Let (u_o,λ_o) be the exact solution of $F(u_o,\lambda_o) = 0$ and w_o the solution of the linear equation $L(\bar{u}_o,\lambda_o)[w] = 0$. Then under suitable conditions for F it follows that

$$||u_o-w_o|| = o(||u_o-\bar{u}_o||) \quad \text{as} \quad ||u_o-\bar{u}_o|| \to 0. \tag{2.14}$$

In our setting, $\bar{u}_o$ represents the computed finite element solution and u_o the exact solution of the given nonlinear problem. Now (2.14) implies that

$$||w_o-\bar{u}_o|| = ||u_o-\bar{u}_o||(1 + o(1)) \quad \text{as} \quad ||u_o-\bar{u}_o|| \to 0 \tag{2.15}$$

and hence, the error $||w_o-\bar{u}_o||$ between the solution of the linearized problem and the computed solution is asymptotically equal to the desired error $||u_o-\bar{u}_o||$. For the approximation of $||w_o-\bar{u}_o||$ we may apply the earlier developed a posteriori error estimates for linear problems.

In order to illustrate the approach we consider the following nonlinear version of the problem (2.1)-(2.2):

$$N[u] \equiv -\frac{\partial}{\partial s} A(\frac{\partial u}{\partial s}) + B(s,u) = C(s,\lambda), \quad 0 < s < 1, \tag{2.16}$$

$$u(0) = u(1) = 0. \tag{2.17}$$

A weak formulation is

$$\int_I [A(u')v' + B(s,u)v - C(s,\lambda)v]ds = 0, \quad \forall\ v \in H_0^1(I)$$

where $D_u B$ denotes the derivative of B with respect to u. Thus, the linearized problem has the form

$$\int_I [A'(u')w'v' + D_u B(s,u)wv - C(s,\lambda)v]ds = 0, \quad \forall\ v \in H_0^1(I). \tag{2.18}$$

If we use again piecewise linear elements on the mesh (2.6), then we have to apply the linear a posteriori estimates to the linearized problem (2.18). More specifically, we have to use the linear problem (2.4) with

$$a(s) = A'(u'(s)), \quad b(s) = D_u B(s,\bar{u}_o), \quad c(s) = C(s,\lambda_o).$$

Hence, if we proceed as in (2.10) then we obtain the error indicators

$$\bar{\eta}_k = \frac{\left| \int_{I_k} [A'(\bar{u}_o')\bar{u}_o' z_k' + B(s,\bar{u}_o)\bar{u}_o z_k - C(s,\lambda_o)]ds \right|}{\left| \int_{I_k} [A'(\bar{u}_o')z_k' z_k' + B(s,\bar{u}_o)z_k z_k]ds \right|^{1/2}} \tag{2.19}$$

where the denominators are assumed to be non-zero. A more complete theory of this approach will be given elsewhere. The approach is exactly the same for problems involving more than one space dimension.

3. THE NFEARS DESIGN

As mentioned already in the Introduction a prototype software system, called NFEARS, is currently under development at the University of Pittsburgh

which has the following characteristics:

(i) The system constitutes an applications-independent finite element solver for a class of two-dimensional, parametrized nonlinear boundary value problems defined by a weak variational formulation.

(ii) Adaptive approaches are employed throughout and the a posteriori error estimates outlined in Section 2 above are used to control the process.

(iii) The system's design is analogous to that of the linear adaptive finite element solver FEARS described in [8], [9], [16] and [17].

Details of the design of NFEARS may be found in [18]. Accordingly we will outline here only some of the principal features. Moreover, since many design aspects of NFEARS correspond to those of FEARS we refer also to the cited references for that system.

The permissible domains are of the same type as in FEARS. In brief, the domain $\bar{\Omega} \subset R^2$ is the union

$$\bar{\Omega} = \bar{\Omega}_1 \cup \bar{\Omega}_2 \cup \ldots \cup \bar{\Omega}_N$$

of finitely many closed, bounded subsets $\Omega_j \subset R^2$ with disjoint, non-empty interiors Ω_j. For each $\bar{\Omega}_j$ a one-to-one, smooth mapping ψ_j onto the unit square $[0,1] \times [0,1]$ of R^2 is given and these mappings satisfy certain natural compatibility conditions (see e.g. [9], [16] or [17]).

The system will have two modules, namely for the solution of problems with one and two unknown functions, respectively. In the case of one unknown function we seek a function u defined on such that

(i) u satisfies prescribed boundary conditions on The form of these conditions is analogous to those of FEARS and includes inhomogeneous Dirichlet conditions, Neumann conditions, as well as mixed boundary conditions.

(ii) u is a stationary point of the functional

$$\int_{\Omega} A(I_1,I_2,\lambda,\mu)dsdt - 2\int_{\Omega} f(s,t,\lambda,\mu)u\ dsdt \tag{3.1}$$

where

$$I_1 = a_1(\nabla u),\quad I_2 = a_2(u)$$

are invariants with respect to rotations of the coordinate system.

Similarly, in the case of two unknown functions we are seeking a vector $u = (u_1,u_2)$ of unknown functions on Ω such that again the above conditions (i) and (ii) hold but with the functional (3.1) replaced by

$$\int_{\Omega} A(I_1,I_2,I_3,\lambda,\mu)dsdt - 2\int_{\Omega} [C_1(s,t,\lambda,\mu)u_1 - C_2(s,t,\lambda,\mu)u_2]dsdt \tag{3.2}$$

where now

$$I_1 = a_1(\nabla u_1,\nabla u_2),\quad I_2 = a_2(\nabla u_1,\nabla u_2),\quad I_3 = a_3(u).$$

This class of problems is fairly general and includes, in particular, most of the basic problems of elasticity theory. Two parameters λ and μ are incorporated in the formulation and hence the equilibrium surface of the problems under study is two-dimensional.

The fundamental computational process is a continuation process which allows for a trace of any path on the surface specified by a given parameter combination with one degree of freedom. More specifically, a form

of the PITCON system described in [19] and [20] is used for this purpose.

The designs of the data structure for the meshes and of the corresponding access algorithms follow essentially those of FEARS except that in NFEARS a biquadratic element will be used. Some details on this and on further aspects of the NFEARS design are given in [18].

4. SOME NUMERICAL EXAMPLES

As a first example, consider the nonlinear boundary value problem (2.16)-(2.17) with

$$A(t) = t/(1+t), \quad B(s,t) = 0, \quad C(s,\lambda) = \lambda \tag{4.1}$$

in which case the exact solution is

$$u(s) = -s + \frac{1}{\lambda} \ln[(e^{\lambda}-1)s + 1], \quad 0 \leq s \leq 1.$$

For growing λ this solution increases rapidly within a small interval near $s = 0$.

A continuation process has been used to compute the solution path with initial point at the origin for $\lambda = 0$. The points with parameter values $\lambda = 1.5$ and $\lambda = 3.0$ were used as target points. At these points, Table 1 gives a comparison of the exact error and the computed a posteriori error estimates. Uniform meshes with degrees of freedom $n = 4,8,16$ were used as well as a nonuniform mesh obtained with the adaptive procedure sketched further below. The table shows that even for comparatively large relative errors, the effectivity of the estimates is excellent. This has been our general experience also with a number of other problems considered so far.

Uniform Meshes, $\lambda = 1.5$, $||u|| = 0.4496$

n	$\|\|e\|\|$	ε	$\|\|e\|\|/\|\|u\|\|\%$	$\varepsilon/\|\|u\|\|\%$	$\varepsilon/\|\|e\|\|\%$
4	0.1657	0.1631	36.86	36.28	98.43
8	0.8760(-1)	0.8710(-1)	19.48	19.37	99.43
16	0.4458(-1)	0.4450(-1)	9.915	9.899	99.84
		$\lambda = 3.0$, $\|\|u\|\| = 1.007$			
4	0.6182	0.5652	61.80	56.50	91.42
8	0.4241	0.3971	42.12	39.43	87.29
16	0.2551	0.2465	25.32	24.47	96.63
		Non-uniform Mesh, $\lambda = 1.5$, $\|\|u\|\| = 0.4496$			
11	0.4911(-1)	0.4908(-1)	10.98	10.98	99.94
		$\lambda = 3.0$, $\|\|u\|\| = 1.007$			
15	0.1417	0.1406	14.07	13.96	99.22

Table 1

As a second example we consider the two-dimensional problem

$$- \frac{\partial}{\partial s} A_1(\nabla u) - \frac{\partial}{\partial t} A_2(\nabla u) = C(s,t,\lambda), \quad \forall\ (s,t) \in \Omega = (0,1) \times (0,1) \tag{4.2}$$

subject to the boundary conditions

$$u = 0 \quad \text{on} \quad \partial\Omega. \tag{4.3}$$

Evidently, in weak form the linearized problem then has the form (2.11) with

$$M(s,t) = \begin{pmatrix} D_1A_1(\nabla u) & D_2A_1(\nabla u) \\ D_1A_2(\nabla u) & D_2A_2(\nabla u) \end{pmatrix}, \quad C(s,t) = 0.$$

Here D_iA_j, $i,j = 1,2$, is the derivative of the coefficient function A_j with respect to its ith-variable. Thus the error indicators and estimates can be computed exactly as stated in Section 2.

As a model case we use the coefficient functions

$$A_1 \equiv (u_s^2+u_t^2)^{\alpha}u_s, \quad A_2 \equiv (u_s^2+u_t^2)^{\alpha}u_t, \quad \frac{1}{2} \leq \alpha \leq 1, \tag{4.4}$$

and choose C in such a way that the exact solution of (4.2)-(4.3) is

$$u = \lambda\, s(1-s)t(1-t), \quad \forall\ s,t \in \bar{\Omega}. \tag{4.5}$$

The continuation process was used to compute the solution path with initial point at the origin for $\lambda = 0$. The points with parameter values $\lambda = 2,4.8,16$ were used as target points. Uniform meshes with $m = 4$ and $m = 16$ elements were used. Table 2 presents again a comparison of the exact error and the

m	$\|\|e\|\|$	ε	$\|\|e\|\|/\|\|u\|\|$%	$\varepsilon/\|\|u\|\|$%	$\varepsilon/\|\|e\|\|$%
$\lambda = 2$	$\|\|u\|\| = 0.2438$				
16	0.5191(-1)	0.5665(-1)	21.29	23.23	109.1
64	0.2551(-1)	0.2615(-1)	10.46	10.73	102.5
$\lambda = 4$	$\|\|u\|\| = 0.6896$				
16	0.1468	0.1625	21.29	23.57	110.7
64	0.7216(-1)	0.7449(-1)	10.46	10.80	103.2
$\lambda = 8$	$\|\|u\|\| = 1.950$				
16	0.4153	0.4726	21.30	24.23	113.8
64	0.2041	0.2137	10.47	10.96	104.7
$\lambda = 16$	$\|\|u\|\| = 5.517$				
16	1.175	1.406	21.29	25.49	119.7
64	0.5773	0.6211	10.46	11.26	107.6

Table 2

computed a posteriori error estimates. As before, the table reflects our experience that the effectivity of the estimates is excellent also in the two-dimensional case even when the errors are large.

It is now widely accepted that for realistic problems it is rarely feasible to construct numerical processes which reliably and effectively achieve a desired accuracy at reasonable cost and yet which do not utilize some form of adaptivity. The design of such an adaptive procedure depends strongly on the goal of the computation (see eg. [7]); but in all cases the availability of reliable error estimators appears to be central to the design of effective adaptive processes.

In nonlinear problems the goal of the computation may take many different forms. For example, it may be required that at each one of the computed points along the solution path the error estimate does not exceed a specified tolerance. In other cases, the goal may be to meet the prescriped error tolerance only at particular target points. In yet other cases, interest may center on the accurate calculation of certain critical points, such as buckling points. Alternately, instead of the computation of solution points with prescribed error behavior we may focus only on the accuracy of the values of a specified functional of these solutions, such as, some stress values, etc.

In nonlinear finite element computations, the principal mechanisms for the control of the adaptive process are the following:

(1) Path Controls

- (a) Steplength selection
- (b) Local parameter selection
- (c) Corrector adjustments (especially in the case of iterative correctors)

(2) Approximation Controls

(a) Mesh-refinement
(b) Mesh-modification
(c) Changes of the order of the element

The path controls are usually part of the continuation process; they affect only the quality of the numerical solution of the current discretized problem. The errors of these solutions with respect to the exact solution of the underlying mathematical model can only be influenced by means of the approximation controls. But there is nevertheless a strong interaction between the two sets of controls which appears not to have been addressed as yet.

There have been only relatively few efforts of incorporating one or several of the approximation controls into nonlinear finite element solvers. Since each one of these three controls has certain advantages and disadvantages, some combination of them may well be desirable. But, even in the linear case the construction of an effective combination of these controls is as yet not fully understood. Accordingly, the current design of NFEARS follows the model of FEARS and utilizes only mesh-refinements for the control of the approximation errors.

Nevertheless, the design of an effective adaptive mesh refinement strategy for maintaining a prescribed error tolerance at all points of the solution path still remains a research problem. We sketch here one such design which appears to be promising. For simplicity, the discussion will be restricted to the one-dimensional problem (2.16)-(2.17).

A mesh partition function is any continuous, strictly monotone-increasing function ϕ: $[0,1] \to [0,1]$ with $\phi(0) = 0$, $\phi(1) = 1$. Then, for any $n \geq 1$, the solutions s_i of the equations

$$\phi(s_i) = \frac{i}{n}, \quad i = 0,1,\ldots,n \tag{4.4}$$

are unique and define a mesh (2.7) on our domain $I = [0,1]$ which will be denoted by $\Delta(\phi,n)$.

For linear problems (2.1)-(2.3) it was shown in [3] that

$$\phi(s) = \frac{1}{\gamma_0}\int_0^s (a(t)u_0(t)^2)^{1/3}dt, \quad s \in I \tag{4.5}$$

with

$$\gamma_0 = \int_0^1 (a(t)u_0(t)^2)^{1/3}dt \tag{4.6}$$

defines an asymptotically optimal partition and that the corresponding errors for the meshes $\Delta_n = \Delta(\phi,n)$ are

$$||e||_E = \frac{\gamma_0^{3/2}}{\sqrt{12}\,n}(1 + O(h_{max})), \text{ as } h_{max} \to 0, \tag{4.7}$$

where h_{max} denotes the maximal mesh step h_i of Δ_n. The asymptotic optimality means that for any other partition function ψ and all n for which the maximal step h_{max} of $\Delta(\psi,n)$ is sufficiently small, the corresponding error is not less thatn (4.7).

Let $\eta_1,\ldots,\eta_{n+1}$ denote the error indicators (2.9) of the finite element solution for a given mesh Δ. Then it turns out that

$$\eta_i^{2/3}h_i = \frac{1}{\sqrt{12}}\left[\int_{I_i} (a(t)u_0(t)^2)^{1/3}dt\right](1 + O(h_{max})), \text{ as } h_{max} \to 0.$$

This can be used to obtain from the computed error indicators $\bar{\eta}_i$ an approximation of the optimal partition function (4.5)-(4.6). Evidently, in

the nonlinear case we may proceed analogously by using again the linearized problem as we did in the computation of the error indicators.

As mentioned before, our aim is to keep the error estimate at each computed point below a given error tolerance. As tolerance it is useful to define the quantity

$$\text{tol} = \delta_{\text{abs}} + \delta_{\text{rel}} \, ||\bar{u}_o||$$

where $||\bar{u}_o||$ is the norm of the computed solution and δ_{abs} and δ_{rel} are absolute and relative error constants. For $\delta_{\text{abs}} = 0$ this means that the relative error has to be less than δ_{rel}, while for $\delta_{\text{rel}} = 0$ the absolute error is not permitted to exceed δ_{abs}.

If at any point along the solution path the computed solution $\bar{u}_o$ has an error estimate below θ tol, where θ, $0 < \theta < 1$, is a given factor, then the process continues with the same mesh. Otherwise the approximate optimal mesh partition function $\tilde{\phi}$ is computed from the error indicators of $\bar{u}_o$ and with it the "ideal" mesh size $\tilde{n}$ as the smallest integer not below $\gamma_o^{3/2}/(\sqrt{12}\,\theta\,\text{tol})$.

This allows us to compute an "ideal" mesh $\tilde{\Delta} = \Delta(\tilde{\phi},\tilde{n})$ and to construct from the current mesh Δ a refined (or de-refined) mesh Δ' which approximates $\tilde{\Delta}$ in some way. There are many ways to accomplish this approximation and a detailed discussion of such techniques shall be given elsewhere. Here we present only some typical results with an adaptive mesh-refinement procedure designed along this line. The sample problem (2.16)-(2.17) with the coefficients (4.1) was used again. For the run given in Table the tolerance was computed with $\delta_{\text{abs}} = 0.01$, $\delta_{\text{rel}} = 0.1$, and the tolerance factor $\theta = 0.75$.

λ	N_{used}	N_{ideal}	error estim	exact error	tolerance	decision
0	4	–	0	0	–	
0.1231	4	4	0.8913(-2)	0.8913(-2)	0.1344(-1)	proceed
0.3395	4	7	0.2516(-1)	0.2517(-1)	0.1949(-1)	refine
	7	7	0.1504(-1)	0.1504(-1)	0.1971(-1)	proceed
0.5551	7	9	0.2440(-1)	0.2441(-1)	0.2594(-1)	refine
	11	9	0.1553(-1)	0.1554(-1)	0.2604(-1)	proceed
0.7655	11	10	0.2133(-1)	0.2134(-1)	0.3323(-1)	proceed
1.15841	11	11	0.3393(-1)	0.3394(-1)	0.4406(-1)	proceed
2.1161	11	15	0.9654(-1)	0.9701(-1)	0.7504(-1)	refine
	15	13	0.5532(-1)	0.5532(-1)	0.7570(-1)	proceed
3.1192	15	21	0.1603	0.1620	0.1146	refine
	18	15	0.9175(-1)	0.9167(-1)	0.1159	proceed

Table 3

Clearly, as expected, whenever a point is accepted and the process continues the error estimate is below the prescribed tolerance.

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